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Stochastic Equity Volatility and the Capital Structure of the Firm [and Discussion]

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Stochastic equity volatility and the capital structure of the firm

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This paper develops a general model for equity volatility when the firm is financed by equity, debt and any other financial instruments like warrants and convertible bonds. The stochastic nature of equity volatility is endogenous and comes from the impact of a change in the value of the firm's assets on the financial leverage. We first present the basic model to value corporate securities, which is an extension of the Black–Scholes model. Then, we are able to propose an analytic approximation for equity volatility, which is shown to be extremely precise. Finally, we study the behaviour of equity volatility when the firm is financed by equity and debt.

1. Introduction

Although the Black–Scholes (1973) model (BSM) is very popular to price equity options and other complex derivative securities, its normative and theoretical requirements are very restrictive. One of the major assumptions needed to derive the BSM is that the value of the underlying asset follows a stationary geometric brownian motion with constant variance. However, for any non-pure equity firm the variance of equity quite probably will not be stationary. Because the volatility of equity is non-stationary, it cannot be used as the underlying asset in BSM for pricing contingent liabilities of the firm or any other derivative. The study of the behaviour of the volatility function for equity will lead to better approximation methods.

In this paper we study the behaviour of equity volatility when its stochastic nature is endogenous and stems from the impact of a change in the value of the firm's assets on the financial leverage. In the next section we present the basic model to value corporate securities. Then, in the third section, we are able to propose an analytic approximation for equity volatility which is shown to be extremely precise. Finally, in the fourth section we look at the behaviour of equity volatility, when the firm is financed by equity and debt.

2. The basic model

Consider a firm whose total asset value $V(t)$ is solution of the Black–Scholes risk-neutral stochastic differential equation,

$$\mathrm{d}V/V = r\,\mathrm{d}t + \bar{\sigma}\,\mathrm{d}z, \tag{1}$$

where r is the continuously compounded risk-free rate of interest which is assumed to be a constant, $\bar{\sigma}^2$ is the instantaneous variance of the return on the asset, which is also assumed to be a constant, t is the current time and z is a standard Wiener

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process. $V(t)$ according to equation (1), is assumed to follow a stationary log-normal distribution with a constant variance such that the discounted price is a martingale. The focus in this paper is on the behaviour of equity volatility when the firm is financed by debt and equity. The model can be extended to more complex capital structures which, for example, include warrants like in Crouhy & Galai (1994) and Bensoussan *et al.* (1993). $V(t)$ is the only exogenous source of risk, but it is generally unobservable and non-traded. Indeed, several components of the firm's capital structure, such as bank loans are not traded on an exchange.

Whenever the assets are financed by equity and additional sources of funds like debt, the stochastic processes for the value of the assets $V(t)$, and of equity $S(t)$, do not coincide. However, developing pricing models in tem of $S(t)$ as the underlying asset does not allow the use of BSM anymore because the stochastic process for $S(t)$ violates the constant volatility assumption.

Our basic hypothesis is that there exists a deterministic functional relation between $S(t)$ and $V(t)$, namely

$$S(t) = S(V(t), t), \tag{2}$$

that implies that $S(t)$ follows a markovian process. Moreover, we assume that there is no arbitrage opportunity so that the function $S(V, t)$ satisfies the traditional Black–Scholes partial differential equation (PDE),

$$-S_t - \tfrac{1}{2}\bar{\sigma}^2 V^2 S_{VV} - rVS_V + rS = 0. \tag{3}$$

To complete the characterization of $S(V, t)$, for $t < T$, it is sufficient to specify the value of $S(V, T)$ at time T. At this stage it is left as a parameter with the condition $S(V, T) \geqslant 0$, so that for any t, $t < T$, $S(V, t) \geqslant 0$ and $\not\equiv 0$.

The only restriction we impose is that we can inverse the function $S(V, t)$, which means that equation $S = S(V, t)$ has a unique solution $V = V(S, t)$. Actually in a levered firm, equity is a monotonic convex function of V, and as a consequence this restriction is satisfied. (In what follows the same notation S or V will be used for an independent argument as well as for the function $S(V, t)$ or $V(S, t)$. Although this may be a source of confusion this preserves the mnemonics.) From the implicit function theorem, we know that the local existence and uniqueness of $V(S, t)$ at any point (V_0, S_0) such that $S_0 = S(V_0, t)$ is guaranteed, provided the derivative $S_V(V_0, t)$ is not equal to zero. In practice, this is not a very stringent assumption since we expect the value of equity to grow with the total asset value. In fact, from (3) the possibility that $S_V(V, t) = 0$ can occur only at isolated points, unless $S(V, t)$ does not depend at all on V. Then, the following identity holds:

$$S \equiv S(V(S, t), t) \quad \text{for all } S \text{ in the domain of } V(S, t), \tag{4}$$

and as a consequence, we also obtain

$$V \equiv V(S(V, t), t), \quad \text{for all } V \text{ in the range of } V(S, t). \tag{5}$$

By differentiating (4) we get

$$V_S(S, t) = 1/S_V(V(S, t), t).$$

From the assumptions concerning the stochastic process of $V(t)$ and using Itô's calculus, it can be shown that the return on equity follows a diffusion described by the following stochastic differential equation

$$\mathrm{d}S/S = r\,\mathrm{d}t + \sigma\,\mathrm{d}z, \tag{6}$$

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where σ denotes the instantaneous volatility of the return on equity, such that

$$\sigma = \sigma(S,t) = \bar{\sigma} S_V(V(S,t),t)\frac{V(S,t)}{S} = \frac{\bar{\sigma}}{S}\frac{V(S,t)}{V_S(S,t)}. \tag{7}$$

Therefore, the value of equity $S(t)$ follows a model similar to $V(t)$ except for its volatility which is no longer a constant, and is precisely given by the function $\sigma(S,t)$ as specified in (7) where the argument S has to be replaced for the value of the process $S(t)$. (For a proof, see Bensoussan *et al.* (1993).)

It is also useful to introduce $\tilde{\sigma}$, the volatility of S as a function of V, while σ denotes the volatility of S as a function of S.

$$\tilde{\sigma} = \tilde{\sigma}(V,t) = \bar{\sigma} S_V(V,t)\, V/S(V,t). \tag{8}$$

We then have the correspondence

$$\begin{cases} \sigma(S,t) = \tilde{\sigma}(V(S,t),t), \\ \tilde{\sigma}(V,t) = \sigma(S(V,t),t). \end{cases} \tag{9}$$

This property will be useful in the next section to derive an analytic approximation for equity volatility as a function of the stock price.

We observe that when V is the underlying variable, $\bar{\sigma}$ the volatility of the return process for V appears in the no-arbitrage equilibrium condition (3) for any asset price $S(V,t)$. If instead we consider S as the underlying variable, the condition for no arbitrage equilibrium for the value function $V(S,t)$ is, as the next lemma shows, the solution of a Black–Scholes type PDE similar to (3), except the volatility term is $\sigma = \sigma(S,t)$, as defined by (7).

Lemma 2.1. *The function $V(S,t)$ defined as the inverse of $S(V,t)$ is solution of the* PDE

$$-V_t - \tfrac{1}{2}\sigma^2 S^2 V_{SS} - rSV_S + rV = 0. \tag{10}$$

Proof. By differentiating equation (4) twice with respect to S, we obtain

$$1 = S_V V_S \quad \text{so that} \quad S_V = 1/V_S,$$

$$0 = V_{SS} S_V + (V_S)^2 S_{VV}, \quad \text{which gives} \quad S_{VV} = -V_{SS}/V_S^3.$$

Then, differentiating (4) with respect to time, it follows that $S_t = -S_V V_t$.

Replacing in equation (3) S_t, S_V and S_{VV} by the above terms and rearranging them, we obtain (10). □

3. Approximating equity volatility

In this section we first show that the functions $\sigma(S,t)$ and $\tilde{\sigma}(V,t)$ are solutions of nonlinear PDEs for which there is no closed-form solution and which are quite complex to solve numerically. However, we are able to derive analytic approximations which are shown to be very accurate. (To the knowledge of the authors, this is the first time that PDEs for the volatility measures (11) and (12), appear in the option pricing literature.)

Proposition 3.1. *$\sigma(S,t)$ satisfies the* PDE,

$$-\sigma_t - \tfrac{1}{2}\sigma^2 S^2 \sigma_{SS} - \sigma_S S(r+\sigma^2) = 0, \tag{11}$$

and $\tilde{\sigma}(V t)$ *is the solution of the* PDE,

$$-\tilde{\sigma}_t-\tfrac{1}{2}\bar{\sigma}^2V^2\tilde{\sigma}_{VV}-\tilde{\sigma}_V\,V(r+\bar{\sigma}\tilde{\sigma})=0. \tag{12}$$

Of course, the boundary values for $\sigma(S,T)$ and $\tilde{\sigma}(V,T)$ have to be specified in order to solve (11) and (12). They follow from (7) and (8) when $V(S,T)$ and $S(V,T)$ are made explicit. The proof is given in Appendix A.

(a) *Approximation of the volatility function*

Equation (11), which defines equity volatility $\sigma(S,t)$, is strongly nonlinear. Not only does it not have any analytic solution, but it is difficult to solve numerically. In this section we propose a very accurate analytical approximation for $\sigma(S,t)$.

We first introduce the functions:

$$\theta(S,t)=S.(\sigma(S,t)-\bar{\sigma}),\quad \tilde{\theta}(V,t)=S(V,t).(\tilde{\sigma}(V,t)-\bar{\sigma}). \tag{13}$$

We next show that θ and $\tilde{\theta}$ are solutions of Black–Scholes PDEs similar to (10) and (3), respectively, although the boundary conditions are obviously different.

Proposition 3.2.

$$-\theta_t-\tfrac{1}{2}\sigma^2S^2\theta_{SS}-\theta_S\,Sr+r\theta=0, \tag{14}$$

$$-\tilde{\theta}_t-\tfrac{1}{2}\bar{\sigma}^2V^2\tilde{\theta}_{VV}-\tilde{\theta}_V\,Vr+r\tilde{\theta}=0. \tag{15}$$

The proof is straightforward.

To simplify (14) further so that we keep only one unknown, θ, instead of two, θ and σ, we replace in (14) σ by its definition (13) in term of θ, i.e.

$$\sigma=\bar{\sigma}+(\theta/S). \tag{16}$$

Now, consider the function $\tilde{\theta}(S,t)$ which should not be confused with $\theta(S,t)$, as by the correspondence law (9), we have:

$$\theta(S,t)=\tilde{\theta}(V(S,t),t)\neq\tilde{\theta}(S,t),$$
$$\tilde{\theta}(S,t)=\theta(S(S,t),t)\neq\theta(S,t).$$

If in (15) we replace the argument V by S, then $\tilde{\theta}(S,t)$ is solution of the equation

$$-\tilde{\theta}_t-\tfrac{1}{2}\bar{\sigma}^2S^2\tilde{\theta}_{SS}-\tilde{\theta}_S\,Sr+r\tilde{\theta}=0. \tag{17}$$

But then (17) can be viewed as (14) after σ has been replaced according to (16) and the term in θ/S has been deleted. This argument provides a first intuition of why $\tilde{\theta}(S,t)$ (solution of PDE 17) might be a good approximation of $\theta(S,t)$. We formally justify this approximation in the next subsection. However, to solve (17) we need to specify its boundary condition at time T, which from (9) and (13) can be written as

$$\tilde{\theta}(S,T)=\theta(S(S,T),T).$$

However, it is generally the case that for any value of S, $S(S,T)\neq S$, so that $\theta(S,T)\neq\theta(S(S,T),T)$. If it is more convenient to compute $\theta(S,T)$, it may be tempting to take as an approximation of $\tilde{\theta}(S,t)$ the solution of (17) with the boundary condition $\theta(S,T)$ instead of $\theta(S,(S,T),T)$. It may happen (this is precisely the case of equity volatility in a firm financed by equity and debt) that $\theta(S,T)$ and $\theta(S(S,T),T)$ coincide although $S\neq S(S,T)$ for all S.

If we take $\tilde{\theta}(S,t)$ as an approximation of $\theta(S,t)$, from (13) the corresponding approximation for $\sigma(S,t)$ is

$$\sigma^*(S,t) = \bar{\sigma} + [\tilde{\theta}(S,t)/S]. \tag{18}$$

This is not $\tilde{\sigma}(S,t)$. In fact, from (13) we obtain

$$\sigma^*(S,t) = \bar{\sigma} + S(S,t)(\tilde{\sigma}(S,t) - \bar{\sigma})/S. \tag{19}$$

Note that $\theta(S,t)$ and $\tilde{\theta}(S,t)$ are not small in general. However, the term $\theta(S,t)/S$ is small for large values of S, in particular when $\theta(S,t)$ is bounded so $\bar{\sigma}$ is already an approximation of $\sigma(S,t)$, but $\sigma^*(S,t)$ is shown to be a much better approximation which is, for practical purposes, extremely accurate.

Indeed, the approximation error for equity volatility is

$$\sigma^*(S,t) - \sigma(S,t) = [\tilde{\theta}(S,t) - \theta(S,t)]/S, \tag{20}$$

and therefore, if the approximation of $\theta(S,t)$ by $\tilde{\theta}(S,t)$ is accurate, the approximation of $\sigma(S,t)$ by $\sigma^*(S,t)$ is even better by the factor $1/S$. On the other hand, the error when we approximate $\sigma(S,t)$ by $\bar{\sigma}$ is $\theta(S,t)/S$, which is much larger than (20).

(*b*) *Justification of the approximation of* $\theta(S,t)$ *by* $\tilde{\theta}(S,t)$

The approximation error is defined by the difference between the approximation and the exact value, i.e.

$$e(S,t) = \tilde{\theta}(S,t) - \theta(S,t)$$

and by the correspondence law (9), we can write $e(S,t)$ as

$$e(S,t) = \tilde{\theta}(S,t) - \tilde{\theta}(V(S,t),t). \tag{21}$$

Theorem 3.1 shows the order of magnitude of the approximation error. To prove this theorem intermediate results are needed and they are shown in Appendix B.

Theorem 3.1. *If there exists a constant a, such that for any constant M,*

$$\left.\begin{aligned} &V(S_V(V,T)-a) \to 0 \quad \text{as} \quad V \to \infty, \\ &|V(S_V(V,T)-a)| \leqslant M, \end{aligned}\right\} \tag{22}$$

and if also

$$[\max(V(S,t),S)]/\min(V(S,t),S) \leqslant k \,\forall\, S \geqslant S_0$$

then the error term $e(S,t)$ *has the property*

$$e(S,t) \to 0 \quad \text{as} \quad S \to \infty.$$

Proof. From expression (21), for the approximation error we can write

$$e(S,t) = (S - V(S,t)) \int_0^1 \tilde{\theta}[V(S) + \lambda(S - V(S)), t]\, d\lambda. \tag{23}$$

Consider a sequence S^n which tends to $+\infty$ as $n \to \infty$. We note from proposition B.2 that for any λ,

$$\tilde{\theta}_V[V(S^n) + \lambda(S^n - V(S^n)), t] \times [V(S^n) + \lambda(S^n - V(S^n))] \to 0 \quad \text{as} \quad n \to \infty$$

and is bounded. Moreover, since we may assume that $S^n \geqslant S_0$, then

$$\left| \frac{[S^n - V(S^n)]}{[V(S^n) + \lambda(S^n - V(S^n))]} \right| \leqslant k.$$

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We can then apply Lebesgue's theorem in the integral (23), giving the conclusion that

$$e(S^n, t) \to 0. \qquad \square$$

In fact the error tends to 0 very fast, as we shall illustrate in the next section.

4. Equity volatility for a firm financed with equity and debt

Consider a firm whose assets, V, are financed by equity, S, and zero-coupon debt, D. The face value of debt is F and it will mature at time T. Following Black & Scholes (1973), Merton (1973) and Galai & Masulis (1976), we can consider the value of equity as a European call option on the assets of the firm, with terminal value at maturity T of debt, $S(V,T) = (V-F)^+$.

Denote by $D(S,t)$ the value of debt at any time $t, t \leqslant T$. Hence, by the correspondence law (9),

$$\tilde{D}(V,t) = D(S(V,t),t).$$

At time T we have

$$\tilde{D}(V,T) = \min(V,F) = V - S(V,T)$$

and, of course, the following balance sheet identity holds.

$$V \equiv S(V,T) + \tilde{D}(V,T).$$

The value of stock, $S(V,t)$, is given by the Black–Scholes formula

$$S(V,t) = BS(V,t \mid F,T,\bar{\sigma},r),$$

where (keeping notations simple by omitting the parameters F, T, $\bar{\sigma}$ and r on which BSM is conditional),

$$\left.\begin{aligned} BS(V,t) &= VN(d_1) - F\,\mathrm{e}^{-r\tau}N(d_2), \\ d_1 \equiv d_1(V) &= \frac{1}{\bar{\sigma}\sqrt{\tau}}\left(\log\left(\frac{V}{F}\right) + (r + \tfrac{1}{2}\bar{\sigma}^2)\,\tau\right), \\ d_2 \equiv d_2(V) &= \frac{1}{\bar{\sigma}\sqrt{\tau}}\left(\log\left(\frac{V}{F}\right) + (r - \tfrac{1}{2}\bar{\sigma}^2)\,\tau\right), \\ \tau &= T - t, \end{aligned}\right\} \qquad (24)$$

and where $N(\,)$ denotes the cumulative standard normal distribution. The Black–Scholes formula is the solution of (3) given the boundary condition $S(V,T) = (V-F)^+$. The derivative of the Black–Scholes formula with respect to the underlying variable V is

$$BS_V(V,t) = N(d_1) \geqslant 0, \qquad (25)$$

so that $S(V,t)$ is strictly monotonic in V for all $t < T$. Hence the function $V(S,t)$ is well defined for $t < T$. Note that at maturity T, if $V > F$, then $S > 0$,

$$V(S,T) = S + F, \quad \text{and} \quad S_V(V,T) = \mathbb{I}_{V \geqslant F}, \qquad (26)$$

where $\mathbb{I}_{\text{condition}}$ is an indicator function which is equal to one when the condition is satisfied, and 0 otherwise. Using formula (41) in Appendix B we get

$$\tilde{\theta}_V(V,t) = \frac{1}{\sqrt{(2\pi\tau)}}\frac{1}{\tau\bar{\sigma}^2}\int_{-\bar{\sigma}\sqrt{(\tau)}d_1(V)}^{+\infty} x \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right)\mathrm{d}x.$$

Hence,

$$\tilde{\theta}_V(V,t) = \frac{1}{\sqrt{(2\pi\tau)}} \exp[-\tfrac{1}{2}(d_1(V))^2],$$

and by using (24) it is also equal to

$$\tilde{\theta}_V(V,t) = \frac{F\mathrm{e}^{-r\tau}}{\sqrt{(2\pi\tau)}} \frac{1}{V} \exp[-\tfrac{1}{2}(d_2(V))^2]. \quad (27)$$

We then see that condition (44) in Appendix B holds, and as a consequence the approximation error $e(S,t)$ tends to zero as S becomes large. In fact, we can derive an explicit expression for the error so that we can assess precisely the error for any value of S. Note that we can write (27) as

$$\tilde{\theta}_V(V,t) = \bar{\sigma} F\,\mathrm{e}^{-r\tau} \frac{\mathrm{d}}{\mathrm{d}V}[N(d_2(V))].$$

Hence

$$\tilde{\theta}(S,t) = \bar{\sigma} F\,\mathrm{e}^{-r\tau} N(d_2(S)), \quad (28)$$

and from (18) the volatility functions $\sigma(S,t)$ will be approximated by

$$\sigma^*(S,t) = \bar{\sigma}\left(1 + \frac{F}{S}\mathrm{e}^{-r\tau} N(d_2(S))\right). \quad (29)$$

The approximation error is

$$e(S,t) = \tilde{\theta}(S,t) - \tilde{\theta}(V(S,t),t) = \bar{\sigma} F\,\mathrm{e}^{-r\tau}\{N(d_2(S)) - N[d_2(V(S,t))]\}. \quad (30)$$

To assess the magnitude of $e(S,t)$ we need bounds on $V(S,t)$. From Black–Scholes' boundary conditions, we have

$$S \leqslant V(S,t) \leqslant S + F\,\mathrm{e}^{-r\tau}. \quad (31)$$

Hence, from (30) we can derive the bounds for the approximation error

$$0 \geqslant e(S,t) \geqslant -\bar{\sigma} F\,\mathrm{e}^{-r\tau}[N(d_2(S + F\,\mathrm{e}^{-r\tau})) - N(d_2(S))].$$

For $S \geqslant F\exp[-(r - \frac{1}{2}\bar{\sigma}^2)\tau]$, we have $d_2(S) \geqslant 0$, and we obtain

$$0 \geqslant e(S,t) \geqslant -[\bar{\sigma} F\,\mathrm{e}^{-r\tau}/\sqrt{(2\pi)}]\exp[-\tfrac{1}{2}(d_2(S))^2][d_2(S + F\,\mathrm{e}^{-r\tau}) - d_2(S)]$$

and, finally,

$$0 \geqslant e(S,t) \geqslant -[F\,\mathrm{e}^{-r\tau}/\sqrt{(2\pi\tau)}]\exp[-\tfrac{1}{2}(d_2(S))^2]\log[1 + (F\,\mathrm{e}^{-r\tau}/S)], \quad (32)$$

which corresponds to an extremely small error. It is smaller than any polynomial function of $1/S$.

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Appendix A. *Proof of Proposition* 3.1

First, we prove that $\sigma(S,t)$ is solution of (11). Let us define $\psi(S,t) \equiv \log V(S,t)$. It then follows from the equity volatility function (7) that

$$\sigma(S,t) = \bar{\sigma}/S\psi_S. \quad (33)$$

From (10) and the definition of $\psi(S,t)$ it follows that

$$-\psi_t - \tfrac{1}{2}S^2\sigma^2(\psi_{SS}+\psi_S^2) - rS\psi_S + r = 0. \tag{34}$$

But from (33)

$$\psi_S = \bar{\sigma}/S\sigma, \quad \psi_{SS} = -\bar{\sigma}(\sigma + S\sigma_S)/S^2\sigma^2.$$

By substituting these expressions in equation (34), and rearranging terms we get

$$\psi_t = -\tfrac{1}{2}[(\bar{\sigma})^2 - \bar{\sigma}\sigma - \bar{\sigma}S\sigma_S] + r - (r\bar{\sigma}/\sigma).$$

Differentiating this last expression with respect to S yields

$$\psi_{tS} = \tfrac{1}{2}\bar{\sigma}\sigma_S + \tfrac{1}{2}\bar{\sigma}(\sigma_S + S\sigma_{SS}) + (r\bar{\sigma}\sigma_S/\sigma^2).$$

But also from (33) the partial differential of ψ_S is

$$\psi_{St} = -\bar{\sigma}\sigma_t/S\sigma^2.$$

Equating these two expressions for ψ_{tS} and ψ_{St} and rearranging terms, we finally obtain (11).

We now turn to the proof that $\tilde{\sigma}(V,t)$ is solution of (12) by following the same line of reasoning as for $\sigma(S,t)$. Define $\phi(V,t) \equiv \log S(V,t)$ and it then follows from (8) that

$$\tilde{\sigma}(V,t) = \bar{\sigma}V\phi_V. \tag{35}$$

From (3) and the definition of $\phi(V,t)$ it can be seen that

$$-\phi_t - \tfrac{1}{2}\bar{\sigma}^2V^2(\phi_{VV}+\phi_V^2) - rV\phi_V + r = 0. \tag{36}$$

But from (35) it follows that

$$\phi_V = \tilde{\sigma}/\bar{\sigma}V, \quad \phi_{VV} = (1/\bar{\sigma})\,[(\tilde{\sigma}_V/V) - (\tilde{\sigma}/V^2)].$$

If we substitute these expressions in (36), after simplifications, we obtain

$$\phi_t = -\tfrac{1}{2}\bar{\sigma}V\tilde{\sigma}_V + \tfrac{1}{2}\bar{\sigma}\tilde{\sigma} - \tfrac{1}{2}\tilde{\sigma}^2 - (r\tilde{\sigma}/\bar{\sigma}) + r.$$

Differentiating this expression with respect to V yields

$$\phi_{tV} = -\tfrac{1}{2}\bar{\sigma}V\tilde{\sigma}_{VV} - (r+\bar{\sigma}\tilde{\sigma})\,\tilde{\sigma}_V/\bar{\sigma}.$$

But, on the other hand, from (35)

$$\phi_{tV} = \tilde{\sigma}_t/\bar{\sigma}V. \qquad \square$$

Equating these two expressions for ϕ_{tV} yields (12).

Appendix B

In what follows we derive intermediate results supporting Theorem 3.1, which helps in assessing the order of magnitude of the approximation error. In Proposition B.1 we derive an explicit formula for $\tilde{\theta}(V,t)$ as a function of the terminal condition $S(V,T)$. Then in Proposition B.2 we derive some important convergence properties. From these results we can prove that the approximation error tends to zero when S becomes large.

Lemma B.1. *The solution of the Black–Scholes PDE (3) is*

$$S(V,t) = \frac{\exp(-r\tau)}{\surd(2\pi\tau)\,\bar{\sigma}}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right)S(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp x, T)\,\mathrm{d}x, \tag{37}$$

where $\tau = T - t$.

The proof follows by directly checking that the function $S(V,t)$, as defined explicitly by (37), is the solution of (3), and takes the value $S(V,T)$ at time T. Note also that by differentiating (37) with respect to V we obtain

$$S_V(V,t) = \frac{1}{\sqrt{(2\pi\tau)}\,\bar{\sigma}} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\tau\bar{\sigma}^2}(x-\tau\bar{\sigma}^2)^2\right] S_V(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp x, T)\,\mathrm{d}x. \tag{38}$$

Then we can derive the following result.

Proposition B.1.

$$\tilde{\theta}(V,t) = \frac{\exp(-r\tau)}{\sqrt{(2\pi\tau)}} \int_{-\infty}^{+\infty} \left(\frac{x}{\tau\sigma^2}-1\right) \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right) S(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp x, T)\,\mathrm{d}x. \tag{39}$$

Proof. From (38) we obtain

$$VS_V(V,t) = V\frac{1}{\sqrt{(2\pi\tau)}\,\bar{\sigma}}\exp(-\tfrac{1}{2}\bar{\sigma}^2\,\tau) \int_{-\infty}^{+\infty} \exp x \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right)$$
$$\times S_V(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp x, T)\,\mathrm{d}x,$$

which can be rewritten as

$$VS_V = \frac{\exp(-r\tau)}{\sqrt{(2\pi\tau)}\,\bar{\sigma}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right) \frac{\mathrm{d}}{\mathrm{d}x} S(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp \mathrm{x}, T)\,\mathrm{d}x.$$

Integrating by parts, we get

$$VS_V = \frac{\exp(-r\tau)}{\sqrt{(2\pi\tau)}\,V\bar{\sigma}} \int_{-\infty}^{+\infty} \frac{x}{\tau\sigma^2} \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right) S(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp x, T)\,\mathrm{d}x,$$

which yields (39), recalling the definitions (13) for $\tilde{\theta}(V,t)$ and (8) for σ. □

By differentiating (39) we obtain

$$\tilde{\theta}_V(V,t) = \frac{\exp(-r\tau)}{\sqrt{(2\pi\tau)}\,V\tau\bar{\sigma}^2} \int_{-\infty}^{+\infty} \left(\frac{x^2}{\tau\sigma^2}-1-x\right) \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right)$$
$$\times S(V\exp[(r-\tfrac{1}{2}\bar{\sigma}^2)\,\tau]\exp x, T)\,\mathrm{d}x. \tag{40}$$

An alternative formula for the derivative $\tilde{\theta}_V(V,t)$ is

$$\tilde{\theta}_V(V,t) = \frac{1}{\sqrt{(2\pi\tau)}}\frac{1}{\tau\bar{\sigma}^2} \int_{-\infty}^{+\infty} x\exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right) S_V(V\exp[\tfrac{1}{2}\bar{\sigma}^2+r]\,\tau)\exp x, T)\,\mathrm{d}x. \tag{41}$$

Formula (41) follows by differentiating the alternative specification for $\tilde{\theta}(V,t)$, namely

$$\tilde{\theta}(V,t) = \frac{\exp[-(r+\tfrac{1}{2}\bar{\sigma}^2)\,\tau]}{\sqrt{(2\pi\tau)}\,\tau\bar{\sigma}^2} \int_{-\infty}^{+\infty} x\exp(-x)\exp\left(-\frac{1}{2}\frac{S^2}{\tau\bar{\sigma}^2}\right)$$
$$\times S(V\exp[(\tfrac{1}{2}\bar{\sigma}^2+r)\,\tau]\exp x, T)\,\mathrm{d}x, \tag{42}$$

which stems from (39) after we operate the change of variable $x \to x+\tau\bar{\sigma}^2$ in the integral. We can now derive convergence properties that are useful in the derivation of Theorem 3.1, where we show that the approximation error tends to zero when S becomes large.

Proposition B. 2. *Suppose there exists a constant a, such that for any constant M,*

$$V(S_V(V,T)-a)\to 0 \quad \text{as} \quad V\to\infty, \qquad |V(S_V(V,T)-a)| \leqslant M. \tag{43}$$

Then, for a convenient function of time $K(t)$

$$V\tilde{\theta}_V(V,t)\to 0 \quad \text{as} \quad V\to\infty, \qquad |V\tilde{\theta}_V(V,t)| \leqslant K(t), \forall\, V. \tag{44}$$

Proof. Using (41) and the fact that we can add any constant a in the integral without changing its value

$$\left(\text{indeed } \int_{-\infty}^{+\infty} x \exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right)\mathrm{d}x = 0\right),$$

we can write

$$V\tilde{\theta}_V(V,t) = \frac{1}{\sqrt{(2\pi\tau)}}\frac{1}{\tau\bar{\sigma}^2}\int_{-\infty}^{+\infty} x\exp\left(-\frac{1}{2}\frac{x^2}{\tau\bar{\sigma}^2}\right) V\{S_V(V\exp[(\tfrac{1}{2}\bar{\sigma}^2+r)\tau]\exp x, T)-a\}\,\mathrm{d}x.$$

Take a sequence $V^n\to\infty$ as $n\to\infty$, then for any fixed x, it follows from assumptions (43) that

$$V^n\{S_V(V^n\exp[(\tfrac{1}{2}\bar{\sigma}^2+r)\tau]\exp x, T)-a\}\to 0.$$

The integrand is in modulus majorized by the fixed integral function

$$\tilde{K}(t)\,|x|\exp(-x)\exp[-\tfrac{1}{2}(x^2/\tau\bar{\sigma}^2)],$$

where $\tilde{K}(t) = M\exp[-(\tfrac{1}{2}\bar{\sigma}^2+r)\tau]$. Hence, (44) follows simply from Lebesgue's theorem. The boundedness property is an immediate consequence of the second of assumptions (43). □

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Discussion

A. D. Wilkie (*Watsons, Reigate, U.K.*). A system in which the share price is known was described, presumably the price for the debt is not known, and the value of the company is not known. It does not seem possible to rely on arbitrage arguments between the price of the share and the value of the company. You concentrate on the

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relations between the standard deviations. But does your argument also not apply to the mean rate of growth of the value of the company? If arbitrage is possible, then the mean rate of growth of the underlying security does not enter the formula for the valuation of an option, and the argument of using an equivalent martingale measure leads one to using the risk free rate. But in this case don't you have to value the share according to the underlying rate of growth of the company, and not at the risk free rate?

M. Crouhy. In our model there is only one source of exogenous uncertainty which is the value of the firm's assets. It drives the dynamics of all the financial claims issued by the firm whether they are equity, debt, convertible bonds or warrants. This source of risk is hedgeable, and therefore it is perfectly legitimate to use the risk neutral valuation framework. In fact by assuming in equation (1), which specifies the stochastic process followed by the assets of the firm, that the drift term is the risk free interest rate, we are already working with the risk-neutral probability measure, Q. Under Q, dz is a brownian motion and discounted security prices are Q-martingales.

M. A. H. Dempster (*University of Essex, U.K.*). It would be interesting to explore the implications of replacing the assumption of a deterministic functional relation $S(t) = S(V(t), t)$, between a realized asset value V of the firm and the corresponding stock price S, by the stochastic differential equation

$$\mathrm{d}S = f(V(t), t) + \mathrm{d}W,$$

where W is an independent Wiener process representing ideosyncratic random effects on the firm's asset value not known to – and hence not valued by – the market. This would make the valuation problems treated partly observed control problems.